ESTIMATION TUTORIAL QUESTIONS

PROBABILITY THEORY

1. The joint probability density function (pdf) $f(x, y)$ is given by

	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$
$\mathbf{x}_1$	0.06	0.07	0.10
$\mathbf{x}_2$	0.09	0.08	0.01
$\mathbf{x}_3$	0.12	0.02	0.09
$\mathbf{x}_4$	0.20	0.10	0.06

   (a) Compute the marginal pdfs $f(x)$ and $f(y)$.

   (b) Compute the conditional pdfs $f(x|y)$ and $f(y|x)$.

   (c) Demonstrate the total probability theorem.

   (d) Demonstrate Bayes theorem.

2. (a) If the pdfs $f(x)$ and $f(z|x)$ are both Gaussian with means $\overline{x}$ and $\overline{z}$, and variances σ_x^2 and σ_z^2 respectively, show that the pdf $f(x|z)$ is also Gaussian. Compute the mean and variance of $f(x|z)$.

   (b) The pdf $f(x, z)$ is jointly Gaussian as

$$\begin{aligned} f(x,z) &= \frac{1}{\sqrt{2\pi}\sigma_x\sigma_z\sqrt{1-\rho^2}} \\ &\times \exp\left[-\frac{1}{2}\frac{(x-\overline{x})^2}{(1-\rho^2)\sigma_x^2} - \frac{1}{2}\frac{(z-\overline{x})^2}{(1-\rho^2)\sigma_z^2} + \rho^2\frac{(x-\overline{x})(z-\overline{z})}{(1-\rho^2)}\right]. \end{aligned}$$

   where ρ is known is the *correlation coefficient*. Compute the mean of the conditional distribution $f(x|z)$. [Hint: Consider only the exponent of the distributions $f(x, z)$ and $f(z)$ (as we know $f(x|z)$ is Gaussian), and use Bayes theorem to compute the exponent of $f(x|z)$. To find the mean, note that the mean is the value of x which makes this exponent a minimum (and the pdf a maximum).]

SYSTEM MODELS

3. (a) Show how an exponentially correlated random variable may be generated from the solution to a first order differential equation driven by white noise. (You are advised to first transfer the problem in to the frequency domain, and at the end to return to the time-domain).

   (b) Write a Matlab function to produce a sequence which is exponentially correlated with time constant T (a first-order Gauss-Markov process). Test your function by computing the sample autocorrelation and power spectral density of your sequence.

   (c) Describe how you might generate a sequence which exhibits second-order correlations (a second-order Gauss-Markov process).

4. Consider the differential equation

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \omega^2 u(t)$$

   where $u(t)$ is some known driving function.

   (a) Write this Equation in state-space form.

   (b) Compute the state transition matrix for this state-space equation.

   (c) Hence, on the assumption that $u(t)$ is approximately constant over the sample interval, write down an equivalent discrete-time state-space equation.

5. A single mechanical gyro is to be used to measure yaw angle attitude of an aircraft. The true yaw rate is integrated to produce an indicated attitude. The indicated attitude then passes through a low-pass filter (modeling the inherent lag in the system) to provide a measurement of true attitude. Write down a state-space description of this sensor and select an appropriate process model to use with this sensor information.

6. Consider again the gyro modeled above. It is discovered that the measurement itself is corrupted by a number of measurement errors:

   (a) A slowly varying bias.

   (b) Wide-band noise of strength Q_1

   (c) Band-limited noise of strength Q_2 and time constant T.

   construct a shaping filter for the gyro that incorporates knowledge of these error sources. Combine this with the system model in question 2 to provide an augmented state-space description for the overall system.

7. (a) Write a **Matlab** programme to simulate the gyro driven by the noise sources described above. To do this, write down the state-space equations for the system and *either* use a modified version of the programme `xtrue.m`, *or* develop the model using **Simulink**. Choose some sensible values for bias, noise strengths and time constant.

   (b) Using data generated by your model, derive an experimental autocorrelation and power spectral density. To do this you will need to make a choice about the length of data and the sampling interval, and to take a sufficiently large average of ACs and PSDs to get sensible experimental plots.

   (c) From your experimental plots, estimate the bias level, noise strengths and time constant in your model. To do this, measure approximate levels and rates from your plots to confirm your input values (do not attempt to fit curves to your data and generate a spectral decomposition).

   (d) Derive expressions for the theoretical autocorrelation and power spectral density. Use **Matlab** to plot these and to compare them to those obtained experimentally.

8. Consider again the gyro system described above. The true yaw attitude is known to be driven by the output of a second order system in the form

$$\ddot{\theta}(t) + 2\zeta\omega_n\dot{\theta}(t)\omega_n^2\theta(t) = \omega_n^2 u(t).$$

   Practically, the input $u(t)$ is a measurement of wind strength which drives the yaw attitude $\theta(t)$ in a harmonic fashion with $\zeta < 1.0$. [This is called the Dryden wind-gust model.] Write down the new state space equations describing the yaw attitude process and the observations that are made of yaw angle.

9. A simple discrete-time model of car motion is:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\phi(t) \\ V(t)\sin\phi(t) \\ \frac{V(t)}{B}\sin\gamma(t) \end{bmatrix},$$

   where x, y, ϕ are the car location and orientation respectively, and $V(t), \gamma(t)$ are the forward velocity and steering angle of the car. Assume that $V(t)$ and $\gamma(t)$ are known perfectly. As the car drives along, it is subject to 'random' disturbances, so that the true vehicle location differs from that that would be computed from knowledge of velocity and steer angle. Write down a linear differential equation, in state-space form, that describes the evolution of the error between true location and the location computed from $V(t)$ and $\gamma(t)$. Discretise this model.

ESTIMATION METHODS

10. Consider the likelihood matrix for a sensor that describes possible target types $\mathbf{x}_i$, given possible observations $\mathbf{z}_i$, $f_1(\mathbf{z}_i \mid \mathbf{x}(j))$:

	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$
$\mathbf{x}_1$	0.4	0.5	0.1
$\mathbf{x}_2$	0.5	0.4	0.1
$\mathbf{x}_3$	0.1	0.1	0.8

Write down the maximum likelihood estimate for x_i. If we now have prior information about target types given by the prior probability vector $f(x) = \{0.6, 0.2, 0.2\}$, compute the maximum *a posterior* estimate for x_i.

11. Consider the measurement

$$z = x + w$$

of the unknown parameter x in the presence of additive measurement noise w, assumed normally distributed with zero mean and variance σ^2. First assume that $\mathbf{x}$ is an unknown constant, then the likelihood function of x is:

$$f(z \mid x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-x)^2}{2\sigma^2}\right)$$

(a) Compute the ML estimate for x.

(b) Compute the MAP estimate for x when the prior is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(x-\overline{x})^2}{2\sigma_0^2}\right)$$

(c) Compute the MAP estimate for x when the prior is given by

$$f(x) = ae^{-ax}$$

12. The equation of a straight line through the origin is given by $y = ax$. N 'observations' x_i and y_i are made of x and y and it is desired to estimate a.

(a) Write down the batch least-squares solution for a.

(b) Write down a recursive least-squares solution for a and b.

(c) How does the problem change if there is an unknown 'bias' b in the equation; $y = ax + b$.

13. Consider the recursive linear estimator for x based on observations $z(i)$:

$$\hat{x}(n) = (1 - \alpha)\hat{x}(n-1) + \alpha z(n)$$

where α is a fixed gain.

(a) Use Matlab to generate a sequence of observations with mean 10.0 and variance 4.0. Compare the estimates of x obtained with $\alpha = 0.1, 0.4, 0.9$.

(b) Show that the estimator is in fact a simple low-pass filter and compute the cut-off frequency as a function of α.

(c) Show that it is possible to generalize a linear estimator to obtain any desired frequency response and explain how it would be implemented.

THE KALMAN FILTER

14. The Kalman filter update equations based on the innovation are

$$\hat{\mathbf{x}}(k \mid k) = \hat{\mathbf{x}}(k \mid k-1) + \mathbf{W}(k)\nu(k)$$

$$\mathbf{P}(k \mid k) = \mathbf{P}(k \mid k-1) - \mathbf{W}(k)\mathbf{S}(k)\mathbf{W}^T(k)$$

   (a) Use these equations directly to deduce that the gain matrix $\mathbf{W}(k)$ that minimizes mean squared error is given by $\mathbf{W}(k) = \mathbf{P}(k \mid k-1)\mathbf{H}^T(k)\mathbf{S}^{-1}(k)$

   (b) Explain the meaning of the term $\mathbf{H}^T(k)\mathbf{S}^{-1}(k)$.

   (c) Explain how these equations are modified if observation and process noises are correlated:

$$\mathrm{E}[\mathbf{v}(k)\mathbf{w}^T(k)] \;= \mathbf{C}(k).$$

15. A particle is moving with constant velocity along the line $y = 20$. It is observed using a bearing-only sensor located at the origin.

   (a) Write down the process and observation equations for this system.

   (b) Find an expression for the extended Kalman filter gain matrix as a function of state prediction and prediction covariance.

   (c) Plot the gain matrix as a function of predicted target bearing. Explain the variation in gain matrix and the effect of low-horizon bearing on estimation accuracy.

16. Consider a feedforward filter for a radar aided inertial system.The two gains for the system are given by $K_1 = \sqrt{2}\omega_n$ and $K_2 = \omega_n^2$.

   (a) Explain the advantages of this 'complimentary' filter in which the accelerometer information is fed forward and the Kalman filter operates only on the error in indicated range.

   (b) Compute the transfer function between measured radar range x_r and estimated range $\hat{x}$, and the transfer function between radar range x_r and estimated velocity $\hat{\dot{x}}$. Use **Matlab** to compute and draw a Bode plot of these transfer functions and explain, in frequency terms, the action the filter has on the estimates based on radar measurements only.

   (c) Compute the transfer function between accelerometer indicated range r_i and estimated range $\hat{r}$. Draw a Bode plot of this transfer function and hence show that the filter acts as a high-pass filter to accelerometer indicated range measurements.

   (d) Explain what effect varying process and observation noise strengths has on filter performance.